

# COLLAPSING IN THE $L^2$ CURVATURE FLOW

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ABSTRACT. We show some results on collapsing phenomena for the  $L^2$  curvature flow, which exhibit a stark difference in the behavior of the flow in the subcritical dimensions  $n = 2, 3$  and supercritical dimensions  $n \geq 5$ . First we show long time existence and convergence of the flow for  $SO(3)$ -invariant initial data on  $S^3$ , as well as an optimal long time existence and convergence statement for three-manifolds with initial  $L^2$  norm of curvature chosen small with respect to the diameter and volume. In the critical dimension  $n = 4$  we show a related low-energy convergence statement with an additional hypothesis. Finally we exhibit some finite time singularities in dimension  $n \geq 5$ , and show examples of finite time singularities in dimension  $n \geq 6$  which are collapsed on the scale of curvature.

## 1. INTRODUCTION

Let  $M^n$  be a smooth compact manifold. Consider the functional of Riemannian metrics

$$(1.1) \quad \mathcal{F}(g) = \int_M |\mathrm{Rm}_g|_g^2 dV_g.$$

This is a natural analogue of the Yang-Mills energy for a Riemannian metric, and studying its negative gradient flow,

$$(1.2) \quad \begin{aligned} \frac{\partial}{\partial t} g &= -\mathrm{grad} \mathcal{F} \\ g(0) &= g_0, \end{aligned}$$

is a natural approach to understanding the structure of this functional. For convenience below, we will call this the  $L^2$  flow. This is a fourth-order, degenerate parabolic equation. Papers on this flow and closely related topics include [4], [20], [26]. Certain obstructions to the long time existence of this flow have by now been established. For instance, curvature blowup at the first singular time was established in [24]. As in the case of Ricci flow, one key difficulty is to understand the possible collapsing behavior at a singular time. Interestingly, it is a simple matter to show that finite time singularities of the flow in dimensions  $n = 2, 3$  *must* be collapsed (see Proposition 3.6 below). The contrapositive statement of Proposition 3.6 is that if one were able to show a noncollapsing result analogous to Perelman's estimate for Ricci flow [16], one immediately concludes the long time existence of solutions to the  $L^2$  flow in dimensions  $n = 2, 3$ . Note that such a statement is plausible on PDE grounds due to the "supercriticality" of the functional  $\mathcal{F}$  in those dimensions, though perhaps counterintuitive due to the highly singular nature of Ricci flow on three-manifolds.

Given the discussion above, let us remark on two cases where the noncollapsing issue for solutions to the  $L^2$  flow is well understood. The first is the case of Riemann surfaces. A compactness/concentration criterion originally due to Chen [8] states roughly that sequences of conformal metrics on Riemann surfaces either converge or experience concentration volume and  $L^1$  concentration of curvature at a point. By fixing a special gauge to reduce the  $L^2$  flow

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to a conformal flow, and by exploiting some energy estimates, we were able to rule out the concentration behavior at finite time to establish long time existence of the  $L^2$  flow on compact Riemann surfaces ([23] Theorem 1).

In higher dimensions the situation is more difficult since the flow cannot be reduced to a conformal flow, and no convenient compactness criteria are available to deal with the collapsing issue. One situation where this difficulty was overcome is related to a certain conformal sphere theorem in four dimensions. In [5], the authors show that a compact Riemannian four-manifold with positive Yamabe constant and sufficiently small  $L^2$  norm of Weyl curvature tensor is smoothly deformable to a spherical space form. In particular, the result yields a classification of the possible diffeotypes satisfying the hypotheses. Moreover, the pinching condition on the Weyl tensor is sharp. Recently, we showed a weaker version of this theorem using the  $L^2$  flow. Specifically, we showed that given a compact Riemannian four-manifold with positive Yamabe constant and sufficiently small  $L^2$  norm of the traceless curvature tensor, the solution to the  $L^2$  flow exists for all time and converges to a spherical space form ([22] Theorem 1). Later this flow result was improved in [4], using a similar method, to yield an explicit value for the required pinching, though this value is still suboptimal.

A critical feature of the flow proofs mentioned in the above paragraph is that the hypotheses of small  $L^2$  norm of Weyl tensor and positive Yamabe constant together imply a bound on the  $L^2$  Sobolev constant. This is a beautiful argument which combines the Gauss-Bonnet theorem and the solution to the Yamabe problem, and is due independently to Gursky [10] and Tian. In the method of [22] and [4] the Sobolev constant bound is used to produce nonflat blowup limits of finite time singularities, and these limits are automatically critical metrics for the corresponding functionals. Then, via a Liouville's Theorem argument one shows that noncompact critical metrics satisfying certain Sobolev constant and  $L^2$  curvature estimates are automatically flat, implying the singularity could not have occurred, thus yielding the long time existence of the flow. The argument of Gursky-Tian perfectly resolves the crucial Sobolev constant issue in the setting of positive Yamabe constant and pinched curvature. Outside of this regime however this remains a difficult problem.

The purpose of this paper is to further flesh out and determine the nature of collapse and singularity formation in the  $L^2$  flow, and moreover to highlight the crucial role played by the dimension of the manifold in understanding this behavior. Our first result is a long-time existence and convergence result for certain warped product 3-manifolds.

**Theorem 1.1.** *Let  $(M^3, g)$  satisfy one of the following conditions:*

- (1)  $M \cong S^3$  and  $g$  is an  $SO(3)$ -invariant metric.
- (2)  $M \cong \Sigma \times S^1$ , where  $\Sigma$  is a compact Riemann surface,  $\chi(\Sigma) \neq 0$ , and  $g = ds^2 + \psi^2 g_\Sigma$ , where  $g_\Sigma$  is a constant curvature metric on  $\Sigma$ , and  $\psi \in C^\infty(S^1, \mathbb{R}_{>0})$ .

*The solution to the volume-normalized  $L^2$  flow with initial condition  $g$  exists on  $[0, \infty)$ , and in the case of  $SO(3)$ -invariant metrics on  $S^3$  converges to a critical metric.*

**Remark 1.2.** The long time existence of  $SO(3)$ -invariant metrics on  $S^3$  is surprising when taken in contrast to the behavior of Ricci flow solutions in this context. Indeed, as shown in [13], neckpinches occur for metrics with such symmetry. Moreover, as established in the work of Perelman [16], [17], neckpinches (degenerate and nondegenerate) are the only finite time local singularities for Ricci flow on three-manifolds. From a PDE perspective though, this long time existence is not so surprising, since the functional  $\mathcal{F}$  is “supercritical” for  $n \leq 3$ , and hence one expects very good existence properties in these dimensions. Indeed, from this perspective it is not so unreasonable to think that solutions to the  $L^2$  flow on three-manifolds always exist for all time (see Conjecture 9.5). The fact that Theorem 1.1 applies to metrics with arbitrary initial energy is encouraging in this respect.

**Remark 1.3.** The proof of the long time existence statements of Theorem 1.1 will hold for the gradient flows of more general functionals. In particular, for any quadratic Riemannian functional satisfying as  $\mathcal{F} \geq \delta \int_M |\text{Rm}|^2$ , the result of Theorem 1.1 should hold.

The next main result is a low energy convergence statement on three-manifolds.

**Theorem 1.4.** *Given  $V > 0, D > 0$ , there exists  $\epsilon > 0$  so that if  $(M^3, g)$  is a compact Riemannian manifold satisfying*

$$(1.3) \quad \begin{aligned} \text{Vol}(g) &\geq V, \\ \text{diam}(g) &\leq D, \\ \mathcal{F}(g) &\leq \epsilon, \end{aligned}$$

*the solution to the volume-normalized  $L^2$  flow exists for all time and converges to a flat metric.*

**Remark 1.5.** Neither the hypothesis of a lower bound on volume nor the hypothesis of the initial bound on diameter can be removed while keeping convergence to a flat metric, making the statement optimal. In particular, on  $S^2 \times S^1$ , the metric  $A g_{S^2} \oplus \frac{1}{A^2} g_{S^1}$  has unit volume and  $\text{diam} = O(A^{\frac{1}{2}})$  and  $\mathcal{F}(g) = O(A^{-2})$  for  $A$  large. Since the universal cover of the manifold is  $S^2 \times \mathbb{R}$ , it cannot admit a flat metric. Indeed, a direct calculation shows that the solution to the  $L^2$  flow with this initial condition satisfies  $A \rightarrow \infty$  as  $t \rightarrow \infty$ . Likewise, the metrics  $g_\epsilon = g_{S^2} \oplus \epsilon^2 g_{S^1}$  have bounded diameter and  $\mathcal{F}(g_\epsilon) = O(\epsilon)$ ,  $\text{Vol}(g_\epsilon) = O(\epsilon)$ .

The key estimate intervening in Theorem 1.4 controls the growth of the first Dirichlet eigenvalue in the presence of a Sobolev constant bound and small global energy. This estimate bears a structural similarity to Perelman's  $\kappa$ -noncollapsing estimate for Ricci flow. Very roughly speaking, Perelman observes that if a solution to the Ricci flow becomes sufficiently collapsed, there are test functions forcing his quantity  $\mu$  to approach  $-\infty$ . Since  $\mu$  is monotonically increasing along the flow, one thereby derives a contradiction. Our approach is similar in that we control the Dirichlet energy  $\mathcal{E}(g, \phi)$  of a test function  $\phi$  along the flow. If at a certain time  $T$  the first Dirichlet eigenvalue is very small, one has a test function  $\phi$  for  $\mathcal{E}$  which yields a very small, positive value. Taking a cue from Perelman's conjugate heat equation, we push this function back to the initial metric by means of the bi-Laplacian heat flow of the time varying metric and derive a test function  $\phi$  such that  $\mathcal{E}(g_0, \phi)$  is again a very small, positive value. Considering the initial first Dirichlet eigenvalue as known, we thus derive an estimate for how fast it can decay along the flow. We emphasize that the proofs really are only similar in their general outline. The energy  $\mathcal{E}$  is *not* monotonic along solutions to the  $L^2$  flow, and the main difficulty is in controlling  $\mathcal{E}$  along the flow. Moreover, we emphasize that we do *not* show a  $\kappa$ -noncollapsing estimate for solutions to the  $L^2$  flow akin to Perelman's estimate for Ricci flow. As already mentioned, such an estimate immediately implies long time existence of the flow on surfaces and three-manifolds (see Proposition 3.6).

By adding an extra hypothesis, we obtain a low energy convergence statement on four-manifolds as well.

**Theorem 1.6.** *Given constants  $A, B > 0$  there exists  $\epsilon(A, B) > 0$  so that if  $(M^4, g)$  is a compact Riemannian manifold with unit volume satisfying*

- $\|\text{grad } \mathcal{F}\|_{L^2} \leq A$
- $C_S(g) \leq B$
- $\int_M |\text{Rm}|^2 \leq \epsilon$

*then the solution to the  $L^2$  flow with initial condition  $g$  exists for all time and converges to a flat metric.*

**Remark 1.7.** It follows from Theorem 1.6 that the difficulty in proving a version of Theorem 1.4 for four-manifolds lies entirely in understanding the short-time behavior of the Sobolev constant. In particular, if one could obtain a doubling-time estimate for the Sobolev constant along the  $L^2$  flow, then by exploiting the gradient property, we conclude the existence of a metric at some time in that time interval with a uniform bound on the Sobolev constant and the  $L^2$  norm of  $\text{grad } \mathcal{F}$ . The analogue of Theorem 1.4 for four-manifolds would then follow, i.e. given energy sufficiently small with respect to the Sobolev constant, the solution to the  $L^2$  flow with this initial condition would exist for all time and converge to a flat metric.

Moving to higher dimensions, we show in § 8 that in all dimensions  $n \geq 5$ , the  $L^2$  flow exhibits finite time singularities, and in dimensions  $n \geq 6$ , the  $L^2$  flow exhibits finite time singularities which do not satisfy an injectivity radius estimate on the scale of curvature. Thus we see quite strikingly the role the dimension plays in understanding the behavior of the  $L^2$  flow, and indeed it seems wise to restrict attention entirely to dimensions  $n \leq 4$ .

Here is an outline of the rest of the paper. In § 2 we give some background on Sobolev and isoperimetric constants, and in § 3 we give background results on the  $L^2$  flow. Section 4 has the proof of Theorem 1.1. In § 5 we derive an estimate for the growth of the first Dirichlet eigenvalue along solutions to the  $L^2$  flow, and we use this in § 6 to prove Theorem 1.4. Section 7 contains the proof of Theorem 1.6. In § 8 we address the behavior of solutions to the  $L^2$  flow in dimensions  $n \geq 5$ . We end in § 9 with a conjectural discussion of the optimal long time existence results for the  $L^2$  flow and their possible applications.

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## 2. ISOPERIMETRIC AND SOBOLEV CONSTANTS

In this section we recall some definitions and theorems related to isoperimetric and Sobolev constants on compact Riemannian manifolds.

**Definition 2.1.** Let  $(M^n, g)$  be a compact Riemannian manifold, and let  $\Omega$  denote a proper open subset of  $M$ . The *isoperimetric constant* is

$$C_I(M, g) := \inf_{\Omega} \frac{\text{Area}(\partial\Omega)}{\min\{\text{Vol}(\Omega), \text{Vol}(M \setminus \Omega)\}^{\frac{n-1}{n}}}.$$

**Definition 2.2.** Let  $(M^n, g)$  be a compact Riemannian manifold. The  $L^1$  *Sobolev constant* of  $M$  is the infimum of all  $C$  such that for any  $f \in C^1(M)$ ,

$$\inf_{\alpha \in \mathbb{R}} \left( \int_M |f - \alpha|^{\frac{n}{n-1}} dV \right)^{\frac{n-1}{n}} \leq C \int_M |\nabla f|^2 dV.$$

**Definition 2.3.** Let  $(M^n, g)$  be a compact Riemannian manifold,  $n \geq 3$ . The  $L^2$  *Sobolev constant*, denoted  $C_S(g)$ , is the infimum of all  $C$  such that for any  $f \in C^1(M)$ ,

$$\left( \int_M f^{\frac{2n}{n-2}} dV \right)^{\frac{n-2}{n}} \leq C \left( \int_M |\nabla f|^2 dV + V^{-\frac{2}{n}} \int_M f^2 dV \right).$$

**Remark 2.4.** The  $L^1$  Sobolev inequality is equivalent to the isoperimetric constant by [3] (see also [6]). Furthermore, by ([15] Lemma 2) an upper bound on the  $L^1$  Sobolev constant implies an upper bound for the  $L^2$  Sobolev constant.

**Theorem 2.5.** Let  $(M^n, g)$  be a complete Riemannian manifold,  $n \geq 3$ . Given  $q \geq 2$ , there exists  $C(q)$  such that for all  $u \in H_1^q(M)$ ,

$$\|u\|_{L^p} \leq C C_S^\alpha \|u\|_{L^m}^{1-\alpha} (\|\nabla u\|_{L^q} + \|u\|_{L^q})^\alpha,$$

where  $2 \leq m \leq p$ ,

$$\alpha = \frac{\frac{1}{m} - \frac{1}{p}}{\frac{1}{m} - \frac{1}{q} + \frac{1}{n}},$$

and

$$\begin{cases} \text{if } q < n, \text{ then } p \leq \frac{nq}{n-q} \text{ and } C = C(n, q), \\ \text{if } q = n, \text{ then } p < \infty \text{ and } C = C(m, p), \\ \text{if } q > n, \text{ then } p \leq \infty \text{ and } C = C(n, m, q). \end{cases}$$

*Proof.* See [14] or [4] for a more recent exposition.  $\square$

**Lemma 2.6.** *Given  $A > 0$  there exists  $D = D(A) > 0$  so that if  $(M^n, g)$  is a compact Riemannian manifold with  $\text{Vol}(g) = 1$  and  $C_S(g) < A$ , then  $\text{diam}(g) < D$ .*

*Proof.* It follows from the argument of ([12] Lemma 3.2) that for  $(M^4, g)$  satisfying the hypotheses, for any  $x \in M$  one has

$$(2.1) \quad \text{Vol}(B_x(1)) \geq \frac{C}{A^n}$$

for a universal constant  $C$ . The result follows from a standard packing argument.  $\square$

**Remark 2.7.** A much more robust argument for this lemma, showing the relationship of Sobolev inequalities and diameter, appears in [1].

**Definition 2.8.** Let  $(M^n, g)$  be a compact Riemannian manifold. Consider the functional

$$\mathcal{E}(\phi, g) := \frac{\|\nabla \phi\|_{L^2}^2}{\|\phi\|_{L^2}^2}.$$

The first Dirichlet eigenvalue of  $g$  is

$$\lambda(g) = \inf_{\{\phi \mid \int_M \phi dV = 0\}} \mathcal{E}(\phi, g).$$

### 3. BACKGROUND ON THE $L^2$ FLOW

In this section we collect some facts about the functional  $\mathcal{F}$  and solutions to the  $L^2$  flow. First we recall ([2] Chapter 4.H) that

$$\text{grad } \mathcal{F} = \delta d \text{ Rc} - \check{R} + \frac{1}{4} |\text{Rm}|^2 g$$

where  $d$  is the exterior derivative induced by the Levi-Civita connection on  $\Lambda^1 \otimes \Lambda^1$ ,  $\delta$  is the  $L^2$ -adjoint of  $d$ , and

$$\check{R}_{ij} = R_i^{pqr} R_{jpqr}.$$

From this it follows that

$$(3.1) \quad \text{tr grad } \mathcal{F} = -\Delta s + \left(\frac{n}{4} - 1\right) |\text{Rm}|^2.$$

Next we note certain elliptic and parabolic coercivity estimates which we will use in the estimates below.

**Lemma 3.1.** ([21] Lemma 2.2) *There exists a universal constant  $C$  such that if  $(M^4, g)$  is a compact four-manifold, then*

$$\|\nabla^2 \text{Rm}\|_{L^2}^2 \leq C \left[ \|\text{grad } \mathcal{F}\|_{L^2}^2 + \int_M |\nabla \text{Rm}|^2 |\text{Rm}| \right].$$

**Proposition 3.2.** ([21] Proposition 4.6) *Given  $A > 0$  there exists  $\epsilon > 0$  so that if  $(M^4, g_t)$  a solution to the  $L^2$  flow on  $[0, T]$  such that*

$$\sup_{t \in [0, T]} C_S(g_t) \leq A,$$

$$\mathcal{F}(g_0) \leq \epsilon,$$

*then*

$$\sup_{[0, T]} \|\text{grad } \mathcal{F}\|_{L^2}^2 + \int_0^T \|\nabla^2 \text{grad } \mathcal{F}\|_{L^2}^2 \leq 2 \|E\|_{L^2(g_0)}^2 + CA^2 \epsilon^{\frac{1}{4}}.$$

Next we recall a fundamental existence result on the existence and long time behavior of solutions to the  $L^2$  flow.

**Theorem 3.3.** ([24] Corollary 1.10) *Let  $(M^n, g)$  be a compact Riemannian manifold. The solution to the  $L^2$  flow with initial condition  $g$  exists on a maximal time interval  $[0, T)$ . Furthermore, if  $T < \infty$  then*

$$\limsup_{t \rightarrow T} |\text{Rm}|_{g(t)} = \infty.$$

**Theorem 3.4.** ([24] Theorem 1.9) *Fix  $m, n \geq 0$ . There exists a constant  $C = C(m, n)$  so that if  $(M^n, g(t))$  is a complete solution to the  $L^2$  flow on  $[0, T]$  satisfying*

$$\sup_{M \times [0, T]} |\text{Rm}| \leq K,$$

*then*

$$(3.2) \quad \sup_{M \times [0, T]} |\nabla^m \text{Rm}| \leq C \left( K + \frac{1}{t^{\frac{1}{2}}} \right)^{1 + \frac{m}{2}}.$$

**Corollary 3.5.** ([24] Corollary 1.5) *Let  $\{(M_i^n, g_i(t), p_i)\}$  be a sequence of complete pointed solutions of the  $L^2$  flow, where  $t \in (\alpha, \omega)$ ,  $-\infty \leq \alpha < \omega \leq \infty$ . Suppose there exists  $K < \infty$  and  $\delta > 0$  such that*

$$\sup_{M_i \times (\alpha, \omega)} |\text{Rm}(g_i)|_{g_i} \leq K, \quad \text{inj}_{g_i(0)}(p_i) \geq \delta.$$

*Then there exists a subsequence  $\{(M_{i_j}, g_{i_j}(t), p_{i_j})\}$  and a one parameter family of Riemannian manifolds  $(M_\infty, g_\infty(t), p_\infty)$  such that  $\{(M_{i_j}, g_{i_j}, p_{i_j})\}$  converges to  $(M_\infty, g_\infty(t), p_\infty)$  in the  $C^\infty$  Cheeger-Gromov topology.*

Next we make the observation mentioned in the introduction, namely that for solutions in dimensions  $n = 2, 3$  any finite time singularity must be collapsed on the scale of curvature.

**Proposition 3.6.** *Let  $(M^n, g(t))$  be a solution to the  $L^2$  flow,  $n = 2, 3$ . Suppose  $g(t)$  exists on a maximal time interval  $[0, T)$ ,  $T < \infty$ . Let  $\{(x_i, t_i)\}$  be a sequence of points such that  $t_i \rightarrow T$  and*

$$|\text{Rm}|(x_i, t_i) = \sup_{[0, t_i]} |\text{Rm}|.$$

*Then*

$$\lim_{i \rightarrow \infty} \text{inj}_g(x_i) |\text{Rm}|(x_i) = 0.$$

*Proof.* Suppose the claim is false, and let  $\{(x_i, t_i)\}$  be the sequence of points as in the statement. Let  $\lambda_i = |\text{Rm}|(x_i, t_i)$ , and let

$$\tilde{g}^i = \lambda_i g \left( t_i + \frac{t}{\lambda_i^2} \right).$$

A direct calculation shows that  $\tilde{g}^i$  is a solution to the  $L^2$  flow on  $[-t_i \lambda_i^2, 0]$  with bounded curvature. Moreover, since we have assumed  $\lim_{i \rightarrow \infty} \text{inj}(x_i) |\text{Rm}|(x_i) > 0$ , we conclude  $\lim_{i \rightarrow \infty} \text{inj}_{\tilde{g}^i}(x_i) > 0$ . It follows from Corollary 3.5 that the sequence of solutions  $\{M, g^i, x_i\}$  contains a subsequence converging to a smooth, nonflat solution to the  $L^2$  flow which we denote  $(M_\infty, g^\infty, x_\infty)$ . But since  $n < 4$ , we note that  $\mathcal{F}(\tilde{g}^i(0)) = \mathcal{F}(\lambda_i g(t_i)) = \lambda_i^{\frac{n}{2}-2} \mathcal{F}(g(0)) \rightarrow 0$ . Thus  $g^\infty$  must be flat, a contradiction.  $\square$

**Remark 3.7.** Note that we need the improved compactness theorem proved recently in ([24] Corollary 1.5) to obtain this result, prior results requiring a global Sobolev constant bound will not suffice here.

We conclude with some remarks on the volume normalized version of the  $L^2$  flow. It follows from (3.1) that if  $(M^n, g(t))$  is a solution to the  $L^2$  flow then

$$\frac{\partial}{\partial t} \text{Vol}(g(t)) = \frac{4-n}{4} \mathcal{F}(g(t)).$$

In particular, the initial value problem

$$(3.3) \quad \begin{aligned} \frac{\partial}{\partial t} g &= -\text{grad } \mathcal{F} + \frac{n-4}{2n} \frac{\mathcal{F}(g)}{\text{Vol}(g)} \cdot g \\ g(0) &= g_0, \end{aligned}$$

preserves the volume of the time dependent metrics, and we call it the *volume-normalized  $L^2$  flow*. One can check that for an initial metric  $g_0$ , the corresponding solutions to the  $L^2$  flow and the volume normalized  $L^2$  flow differ by a rescaling in space and time. Furthermore, the volume normalized  $L^2$  flow is the gradient flow of the functional

$$(3.4) \quad \tilde{\mathcal{F}}(g) = \text{Vol}(g)^{\frac{4-n}{n}} \mathcal{F}(g)$$

**Definition 3.8.** A solution  $(M^n, g(t))$  to the volume normalized  $L^2$  flow is *nonsingular* if it exists on  $[0, \infty)$  with a uniform bound on the curvature tensor.

**Theorem 3.9.** *Let  $(M^n, g(t))$  be a nonsingular solution to the volume normalized  $L^2$  flow. Then either*

- *For all  $p \in M$ ,  $\limsup_{t \rightarrow \infty} \text{inj}_p g(t) = 0$ .*
- *There exists a sequence of times  $t_i \rightarrow \infty$  such that  $\{g(t_i)\}$  converges to a smooth metric on  $M$  which is critical for  $\tilde{\mathcal{F}}$ .*
- *There exists a sequence of points  $(p_i, t_i), t_i \rightarrow \infty$  such that  $\{(M, g(t_i), p_i)\}$  converges to a complete noncompact finite volume metric which is critical for  $\tilde{\mathcal{F}}$ .*

#### 4. THREE-MANIFOLDS WITH SYMMETRY

In this section we study solutions to the  $L^2$  flow where the initial condition is either a warped product of a surface with constant curvature over  $S^1$ , or an  $SO(3)$ -invariant metric on  $S^3$ . These two cases are natural to combine since  $SO(3)$ -invariant metrics on  $S^3$  are equivalent to warped product metrics on  $(-1, 1) \times S^2$  with certain boundary conditions prescribed below. Specifically, fix  $\Sigma$  a compact Riemann surface,  $\chi(\Sigma) \neq 0$ , and let  $g_\Sigma$  denote a metric of constant curvature



$-1, 1$  depending on the Euler characteristic of  $\Sigma$ . Let  $M \cong (-1, 1) \times \Sigma$ . Let  $\phi, \psi : (-1, 1) \rightarrow \mathbb{R}_{>0}$ , and consider the Riemannian metric

$$(4.1) \quad g = \phi(x)^2 dx^2 + \psi(x)^2 g_\Sigma.$$

It will frequently be useful to use the natural geometric coordinate of lateral distance from the slice  $\Sigma \times \{0\}$ . In particular, set

$$(4.2) \quad s(x) = \int_0^x \phi(w) dw.$$

The range of  $s$  is some interval we will always refer to as  $(a, b)$ . Moreover, let

$$L(g) := \int_{-1}^1 \phi(x) dx.$$

$L$  represents the length of the base circle in the case of a product topology, or the distance from the north pole to the south pole in the case of  $SO(3)$  invariant metrics on  $S^3$ . Of course  $L = b - a$ . Using the parameter  $s$ , the metric (4.1) takes the form

$$(4.3) \quad g = ds^2 + \psi(s)^2 g_\Sigma.$$

If we impose periodic boundary conditions, i.e. that  $\phi$  and  $\psi$  extend to smooth functions on  $S^1$ , then  $g$  defines a metric on  $S^1 \times \Sigma$ . To produce  $SO(3)$ -invariant metrics on  $S^3$ , we will impose

$$(4.4) \quad \begin{aligned} \lim_{x \rightarrow \pm 1} \psi &= 0 \\ \lim_{x \rightarrow \pm 1} \psi_s &= \mp 1. \end{aligned}$$

Next we establish some geometric estimates for the Riemannian manifolds described above. We will informally refer to these as warped products, with the assumption that the boundary conditions have been chosen appropriate to the given topology.

**Lemma 4.1.** *Let  $(M^3, g)$  be a warped product. Let  $v$  and  $w$  denote vectors in  $\pi^*T\Sigma$ . Then*

$$(4.5) \quad K_1 = K \left( \frac{\partial}{\partial s} \wedge v \right) = -\frac{\psi_{ss}}{\psi}, \quad K_2 = K(v \wedge w) = \frac{K_\Sigma - \psi_s^2}{\psi^2}.$$

**Lemma 4.2.** *Let  $(M^3, g)$  be a warped product. Then*

$$(4.6) \quad \text{Vol}(g) = \text{Vol}(g_\Sigma) \int_a^b \psi^2 dw,$$

$$(4.7) \quad \int_M |\text{Rm}|^2 dV = \text{Vol}(g_\Sigma) \int_a^b \left( 4\psi_{ss}^2 + 2 \frac{(K_\Sigma - \psi_s^2)^2}{\psi^2} \right) dw.$$

*Proof.* We directly compute

$$\text{Vol}(g) = \int_M dV_g = \int_a^b \int_\Sigma \psi^2 d\Sigma dw = \text{Vol}(g_\Sigma) \int_a^b \psi^2 dw.$$

Next, using (4.5) we compute

$$\begin{aligned} \int_M |\text{Rm}|^2 dV &= \int_a^b \int_\Sigma \left( 4 \frac{\psi_{ss}^2}{\psi^2} + 2 \frac{(K_\Sigma - \psi_s^2)^2}{\psi^4} \right) \psi^2 d\Sigma dw \\ &= \text{Vol}(g_\Sigma) \int_a^b \left( 4\psi_{ss}^2 + 2 \frac{(K_\Sigma - \psi_s^2)^2}{\psi^2} \right) dw. \end{aligned}$$

□



**Lemma 4.3.** *There is a constant  $\delta > 0$  so that if  $(S^3, g)$  is an  $SO(3)$ -invariant metric with  $|\text{Rm}| \leq 1$ , then  $L \geq \delta$ .*

*Proof.* Recall that our metric satisfies (4.4). Let  $\mu = \sup\{s > a | \frac{1}{2} \leq \psi_s \leq 2\}$ . Clearly  $L \geq \mu$ , thus it suffices to bound  $\mu$  from below. First note that on  $[0, \mu]$  certainly  $\psi \leq 2\mu$ . Without loss of generality assume  $\mu \leq \frac{1}{8}$ , so that  $\psi \leq \frac{1}{4}$  on  $[0, \mu]$ . Now observe that since  $|\text{Rm}| \leq 1$ ,

$$\psi_{ss} = -\psi K_1 \geq -\psi \geq -\frac{1}{4}.$$

It follows that, on  $[0, \mu]$ ,  $\psi_s \geq 1 - \frac{\mu}{4}$ . Likewise we can estimate on  $[0, \mu]$

$$\psi_s^2 = K_\Sigma - K_2 \psi^2 \leq 1 + \psi^2 \leq 1 + (2\mu)^2 \leq 1 + \frac{1}{4}.$$

This implies a lower bound for  $\mu$ , and the lemma follows.  $\square$

**Proposition 4.4.** *Let  $(M^3, g)$  be a warped product with fiber  $\Sigma$ ,  $\chi(\Sigma) \neq 0$ , further satisfying  $|\text{Rm}| \leq 1$ . Given  $\epsilon > 0$ , there exists  $\delta > 0$  so that if  $L_g \geq \epsilon$  then*

$$(4.8) \quad \text{inj}_g \geq \delta.$$

*Proof.* We first consider the case of product topology, i.e.  $M \cong S^1 \times \Sigma$ . Fix  $s_0 \in (a, b)$  a minimum point for  $\psi$ . At this point we compute

$$1 \geq |K_2| = \left| \frac{K_\Sigma - \psi_s^2}{\psi^2} \right| = \frac{|K_\Sigma|}{\psi^2}.$$

We conclude that for all  $s$ ,

$$\psi(s) \geq \psi(s_0) \geq |K_\Sigma| = 1$$

since  $\chi(\Sigma) \neq 0$ . We now show a lower volume growth estimate for sufficiently small balls. Fix a constant  $r_0 > 0$  so that  $r_0 \leq \text{inj}(g_\Sigma)$ , and also  $r_0 \leq \frac{\epsilon}{2} \leq \frac{L}{2}$ . Now fix an arbitrary  $(p_0, s_0) \in \Sigma \times S^1$  and consider  $B_r(p_0, s_0)$ . We want to show that there is a uniform constant  $\mu > 0$  so that, for all  $r \leq r_0$ ,

$$\frac{\text{Vol}(B_r(p_0, s_0))}{r^3} \geq \mu.$$

Without loss of generality we can reparameterize  $s$  so that  $s_0 = 0$ , and the range of  $s$  is  $(-\frac{L}{2}, \frac{L}{2})$ . First we claim the inclusion

$$(4.9) \quad U := B_{\frac{r}{2}, \psi^2(s_0)g_\Sigma}(p_0) \times \left[ s_0 - \frac{r}{2}, s_0 + \frac{r}{2} \right] \subset B_{r,g}(p_0, s_0).$$

To show this let  $(q, t) \in U$  and let  $\gamma$  denote the curve which is the concatenation of the shortest geodesic in  $\Sigma$  connecting  $p_0$  and  $q$ , in the metric  $\psi^2(s_0)g_\Sigma$ , with the lateral curve connecting  $(q, s_0)$  to  $(q, t)$ . One has

$$d_g((p_0, s_0), (q, t)) \leq \text{Length}(\gamma) \leq \frac{r}{2} + \frac{r}{2} = r.$$

Therefore the inclusion (4.9) holds. One can then compute

$$\begin{aligned}
\text{Vol}(B_r(p_0, s_0)) &\geq \text{Vol}(U) \\
&= \int_{s_0 - \frac{r}{2}}^{s_0 + \frac{r}{2}} \int_{B_{\frac{r}{2}, \psi^2(s_0)g_\Sigma}(p_0, s_0)} \psi^2(w) d\Sigma dw \\
&\geq \inf \psi^2 \int_{s_0 - \frac{r}{2}}^{s_0 + \frac{r}{2}} \int_{B_{\frac{r}{2}, \psi^2(s_0)g_\Sigma}(p_0, s_0)} d\Sigma dw \\
&\geq \frac{\nu}{2} r^3.
\end{aligned}$$

where  $\nu$  is a lower bound on the volume growth of  $g_\Sigma$ . The lower bound on volume growth follows, and by Cheeger's lemma the proposition follows. This argument can be repeated for  $SO(3)$ -invariant metrics on  $S^3$ .  $\square$

*Proof of Theorem 1.1.* Let  $(M^3, g(t))$  be a solution to the volume normalized  $L^2$  flow as in the statement. The first step is to show long time existence. We know from Theorem 3.3 that if the maximal existence time is  $T < \infty$  then

$$\lim_{t \rightarrow T} |\text{Rm}|_{g(t)} = \infty.$$

Choose a sequence of points  $(x_i, t_i)$  such that

$$\lambda_i := |\text{Rm}(x_i)|_{g(t_i)} = \sup_{M \times [0, t_i]} |\text{Rm}|.$$

Let

$$g_i(t) := \lambda_i g \left( t_i + \frac{t}{\lambda_i^2} \right)$$

By construction one notes that the solution  $g_i(t)$  exists on  $[-\lambda_i^2 t_i, 0]$ , and moreover

$$(4.10) \quad \sup_{M \times [-\lambda_i^2 t_i, 0]} |\text{Rm}| = |\text{Rm}(x_i)|_{g_i(0)} = 1.$$

We want to take a convergent subsequence of these solutions  $\tilde{g}_i$ , and to obtain a manifold in the limit we require a lower bound on the injectivity radius. By Proposition 4.4 it suffices to show a lower bound on the lateral distance  $L$ .

Here we break into cases. In the case of  $SO(3)$ -invariant metrics on  $S^3$ , Lemma 4.3 provides the required lower bound. For the remaining cases we argue by contradiction. Assume  $L_i \rightarrow 0$ . Note that the solutions  $(M, g_i(t))$  exist, for sufficiently large  $i$ , on  $[-1, 0]$ . It follows from (4.10) and Theorem 3.4 that there is a uniform constant  $C$  such that

$$\sup_M |\nabla \text{Rm}|_{g_i(0)} \leq C.$$

Say the point  $x_i$  is given by  $(p_i, s_i) \in \Sigma \times (a, b)$ . By integrating along a lateral geodesic one concludes

$$|\text{Rm}|_{g_i(0)}(p_i, s_i \pm \tau) \geq |\text{Rm}|_{g_i(0)}(p_i, s_i) - \tau C \geq 1 - CL_i.$$

Since the curvatures are functions of the parameter  $s$  only, and since  $L_i \rightarrow 0$  we may conclude that for sufficiently large  $i$  one has

$$(4.11) \quad \inf_M |\text{Rm}|_{g_i(0)} \geq \frac{1}{2}.$$

Since the volume of the unscaled metrics was fixed, for the rescaled solutions it follows that

$$(4.12) \quad \text{Vol}(g_i(0)) \geq \text{Vol}(g(0))\lambda_i^{\frac{3}{2}}.$$

Combining (4.11) and (4.12) we see that

$$(4.13) \quad \int_M |\text{Rm}|_{g_i(0)}^2 dV_{g_i(0)} \geq \frac{\text{Vol}(g(0))}{4} \lambda_i^{\frac{3}{2}} \rightarrow \infty$$

as  $i \rightarrow \infty$ . But of course  $\int_M |\text{Rm}|^2(g(t))dV \leq C$  thus

$$\int_M |\text{Rm}|_{g_i(0)}^2 dV_{g_i(0)} = \lambda_i^{-\frac{1}{2}} \int_M |\text{Rm}|_{g(t_i)}^2 dV_{g(t_i)} \rightarrow 0,$$

contradicting (4.13). It follows that there is a uniform constant  $\mu > 0$  so that

$$L(g_i(0)) \geq \mu > 0.$$

We now conclude from Proposition 4.4 that there is a constant  $\delta > 0$  independent of  $i$  so that

$$\text{inj}_{g_i(0)}(x_i) > \delta.$$

Using this and (4.10), we conclude from Theorem 3.5 that there is a subsequence of  $\{(M, g_i(t), x_i)\}$  converging to a pointed solution  $(M_\infty^3, g_\infty(t), x_\infty)$  to the volume normalized  $L^2$  flow. By construction this solution satisfies

$$(4.14) \quad |\text{Rm}|_{g_\infty(0)}(x_\infty) = 1.$$

However, one has

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_M |\text{Rm}|_{g_i(0)}^2 dV_{g_i(0)} &= \lim_{i \rightarrow \infty} \lambda_i^{-\frac{1}{2}} \int_M |\text{Rm}|_{g(t_i)}^2 dV_{g(t_i)} \\ &\leq \lim_{i \rightarrow \infty} \lambda_i^{-\frac{1}{2}} \int_M |\text{Rm}|_{g(0)}^2 dV_{g(0)} \\ &= 0. \end{aligned}$$

It follows by Fatou's Lemma that

$$\int_{M_\infty} |\text{Rm}|_{g_\infty(0)}^2 dV_{g_\infty(0)} = 0,$$

contradicting (4.14). It follows that the curvature is bounded on finite time intervals, and therefore the solution exists on  $[0, \infty)$ . To show the uniform curvature bound, one can repeat the argument by contradiction above, again blowing up around a sequence of points realizing the spacetime maximum of curvature.

Turning now to the convergence statement for  $SO(3)$ -invariant metrics, we note that we have shown that the solutions under consideration are nonsingular in the sense of Definition 3.8. We must consider the different possibilities for the limiting behavior given by Theorem 3.9. First let us globally rescale the solution to (3.3) so that

$$\sup_{M \times [0, \infty)} |\text{Rm}| \leq 1.$$

Using Lemma 4.3 and Proposition 4.4 we conclude that

$$\inf_{M \times [0, \infty)} \text{inj}_g \geq \delta > 0.$$

It follows that the first and third possibilities of Theorem 3.9 are impossible, therefore we must have subsequential convergence to a critical metric.  $\square$

## 5. ESTIMATES OF FIRST DIRICHLET EIGENVALUE

The main purpose of this section is to prove Theorem 5.1 below, which is an estimate on the decay of the first Dirichlet eigenvalue of the evolving metric along a solution to the  $L^2$  flow. The strategy of the proof is to take a test function for the functional  $\mathcal{E}$  at a certain forward time in the flow, then push it back to the initial time using the backwards biharmonic heat flow.

**Theorem 5.1.** *There exist universal constants  $C > 0$ ,  $\bar{\lambda} > 0$  such that given  $A > 0$ , there exists  $\epsilon > 0$ , so that if  $(M^n, g_t)$  is a solution to the  $L^2$  flow on a compact manifold  $M^n$ ,  $n \leq 4$ , on a time interval  $[0, T]$ ,  $T \leq 1$ , satisfying*

- (1)  $\mathcal{F}(g_0) \leq \epsilon$
- (2) For all  $f \in C^1(M)$ , all  $t \in [0, T]$ ,  $\|f\|_{L^4}^2 \leq A \left( \|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2 \right)$
- (3)  $\lambda(g_T) \leq \bar{\lambda}$ .

Then

$$\lambda(g_0) \leq 2\lambda(g_T) + CA^2\epsilon^{\frac{1}{2}}.$$

**Remark 5.2.** The second hypothesis above is of course just the usual  $L^2$  Sobolev inequality when  $n = 4$ , but we have restated it here to unify the discussions for all dimensions  $n \leq 4$ , which will simplify the proof below.

Without further comment we fix throughout this section a solution  $(M^n, g_t)$  satisfying the hypotheses above, the notation of Theorem 5.1, and the notation in the following paragraph. As a notational convenience we set

$$E := \frac{\partial}{\partial t} g.$$

Furthermore, let  $\tau = T - t$  and define  $\phi$  on  $M \times [0, T]$  via

$$(5.1) \quad \begin{aligned} \frac{\partial}{\partial \tau} \phi &= -\Delta_{g_\tau}^2 \phi + \frac{1}{2} \phi \operatorname{tr}_g E \\ \phi(T) &= \phi_T. \end{aligned}$$

To clarify, as written the first equation above is parabolic in the backwards time parameter  $\tau$ , therefore we specify a *final* value and solve backwards in time. As this is a linear parabolic equation defined with respect to a smooth one-parameter family of metrics, the existence of  $\phi^i$  on the whole interval  $[0, T]$  follows from standard estimates for linear parabolic equations. In deriving the estimates below we adopt the convention that  $C$  always denotes a universal constant, which may change from line to line. First we observe that the  $L^1$  norm of  $\phi$  is fixed for all times, which is the purpose of inserting the zeroth order term into (5.1).

**Lemma 5.3.** *One has*

$$\frac{\partial}{\partial \tau} \int_M \phi dV = 0$$

*Proof.* We directly compute

$$\frac{\partial}{\partial \tau} \int_M \phi dV = \int_M \left( \frac{\partial}{\partial \tau} \phi - \phi \frac{1}{2} \operatorname{tr}_g E \right) dV = \int_M -\Delta^2 \phi dV = 0.$$

□

In the next two lemmas we derive differential inequalities for the evolutions of  $\|\phi\|_{L^2}$  and  $\|\nabla \phi\|_{L^2}^2$ .

**Lemma 5.4.** *One has*

$$\frac{\partial}{\partial \tau} \|\phi\|_{L^2}^2 = -2 \|\Delta \phi\|_{L^2}^2 + \int_M \left[ \frac{1}{2} \operatorname{tr}_g E \phi^2 \right] dV$$

*Proof.* We compute

$$\begin{aligned} \frac{\partial}{\partial \tau} \|\phi\|_{L^2}^2 &= 2 \int_M \phi \left( -\Delta^2 \phi + \phi \frac{1}{2} \operatorname{tr}_g E \right) + \int_M \left[ -\frac{1}{2} \operatorname{tr}_g E \phi^2 \right] dV \\ &= -2 \|\Delta \phi\|_{L^2}^2 + \int_M \left[ \frac{1}{2} \operatorname{tr}_g E \phi^2 \right] dV. \end{aligned}$$

□

**Lemma 5.5.** *There exists a universal constant  $C$  such that*

$$(5.2) \quad \frac{\partial}{\partial \tau} \|\nabla \phi\|_{L^2}^2 = -2 \|\nabla \Delta \phi\|_{L^2}^2 - \int_M \left[ \langle E, \nabla \phi \otimes \nabla \phi \rangle + \operatorname{tr}_g E \left( \phi \Delta \phi + \frac{1}{2} |\nabla \phi|^2 \right) \right] dV.$$

*Proof.* A direct calculation shows that

$$\begin{aligned} \frac{\partial}{\partial \tau} \|\nabla \phi\|_{L^2}^2 &= \int_M \left[ \langle -E, \nabla \phi \otimes \nabla \phi \rangle + 2 \left\langle \nabla \left( -\Delta^2 \phi + \frac{1}{2} \phi \operatorname{tr}_g E \right), \nabla \phi \right\rangle - \frac{1}{2} \operatorname{tr}_g E |\nabla \phi|^2 \right] dV \\ &= -2 \|\nabla \Delta \phi\|_{L^2}^2 - \int_M \left[ \langle E, \nabla \phi \otimes \nabla \phi \rangle + \operatorname{tr}_g E \left( \phi \Delta \phi + \frac{1}{2} |\nabla \phi|^2 \right) \right] dV. \end{aligned}$$

□

We are ready now to derive a fundamental differential inequality for the evolution of  $\mathcal{E}$ .

**Proposition 5.6.** *There exists a universal constant  $C$  such that if  $\epsilon \leq \frac{1}{2A}$ , one has*

$$(5.3) \quad \begin{aligned} \frac{\partial}{\partial \tau} \mathcal{E}(g, \phi) &\leq - \frac{\|\nabla^3 \phi\|_{L^2}^2}{\|\phi\|_{L^2}^2} \\ &\quad + C \left[ \mathcal{E}^3 + A \|E\|_{L^2} \mathcal{E}^2 + \left( A^2 \|E\|_{L^2}^2 + A \|E\|_{L^2} \right) (1 + \mathcal{E}) \right]. \end{aligned}$$

*Proof.* Applying Lemmas 5.4 and 5.5 yields

$$\begin{aligned} \frac{\partial}{\partial \tau} \mathcal{E}(\phi_t, g_t) &= -2 \frac{\|\nabla \Delta \phi\|_{L^2}^2}{\|\phi\|_{L^2}^2} - \frac{\int_M \left[ \langle E, \nabla \phi \otimes \nabla \phi \rangle + \operatorname{tr}_g E \left( \phi \Delta \phi + \frac{1}{2} |\nabla \phi|^2 \right) \right] dV}{\|\phi\|_{L^2}^2} \\ &\quad + 2 \frac{\|\nabla \phi\|_{L^2}^2 \|\Delta \phi\|_{L^2}^2}{\|\phi\|_{L^2}^4} - \frac{\|\nabla \phi\|_{L^2}^2 \int_M \left[ \frac{1}{2} \operatorname{tr}_g E \phi^2 \right] dV}{\|\phi\|_{L^2}^4}. \end{aligned}$$

We need to estimate each of these terms. First we have

$$\begin{aligned} \left| \int_M \left[ \langle -E, \nabla \phi \otimes \nabla \phi \rangle + \frac{1}{2} \operatorname{tr}_g E |\nabla \phi|^2 \right] dV \right| &\leq C \int_M |E| |\nabla \phi|^2 \\ &\leq C \|E\|_{L^2} \|\nabla \phi\|_{L^4}^2 \\ &\leq CA \|E\|_{L^2} \left( \|\nabla^2 \phi\|_{L^2}^2 + \|\nabla \phi\|_{L^2}^2 \right). \end{aligned}$$

Integrating by parts and applying the Sobolev inequality yields

$$\|\nabla^2 \phi\|_{L^2}^2 \leq \|\nabla \phi\|_{L^2} \|\nabla \Delta \phi\|_{L^2} + A \|\operatorname{Rm}\|_{L^2} \left( \|\nabla^2 \phi\|_{L^2}^2 + \|\nabla \phi\|_{L^2}^2 \right).$$

Choosing  $\|\text{Rm}\|_{L^2} \leq \frac{1}{2A}$  yields

$$\|\nabla^2 \phi\|_{L^2}^2 \leq C \left( \|\nabla \phi\|_{L^2} \|\nabla \Delta \phi\|_{L^2} + \|\nabla \phi\|_{L^2}^2 \right),$$

and hence

$$\begin{aligned} \left| \int_M \left[ \langle -E, \nabla \phi \otimes \nabla \phi \rangle + \frac{1}{2} \text{tr}_g E |\nabla \phi|^2 \right] dV \right| &\leq CA \|E\|_{L^2} \left( \|\nabla \phi\|_{L^2} \|\nabla \Delta \phi\|_{L^2} + \|\nabla \phi\|_{L^2}^2 \right) \\ &\leq \frac{1}{3} \|\nabla \Delta \phi\|_{L^2}^2 + C \left( A^2 \|E\|_{L^2}^2 + A \|E\|_{L^2} \right) \|\nabla \phi\|_{L^2}^2. \end{aligned}$$

Next we have

$$\begin{aligned} \int_M \text{tr}_g E \phi \Delta \phi &\leq \|E\|_{L^2} \|\phi\|_{L^4} \|\Delta \phi\|_{L^4} \\ &\leq \|E\|_{L^2} A (\|\nabla \phi\|_{L^2} + \|\phi\|_{L^2}) (\|\nabla \Delta \phi\|_{L^2} + \|\Delta \phi\|_{L^2}) \\ &\leq C \|E\|_{L^2} A (\|\nabla \phi\|_{L^2} + \|\phi\|_{L^2}) (\|\nabla \Delta \phi\|_{L^2} + \|\nabla \phi\|_{L^2}) \\ &\leq \frac{1}{3} \|\nabla \Delta \phi\|_{L^2}^2 + CA \|E\|_{L^2} (1 + A \|E\|_{L^2}) (\|\nabla \phi\|_{L^2}^2 + \|\phi\|_{L^2}^2) \end{aligned}$$

Also, we can estimate

$$\begin{aligned} 2 \frac{\|\nabla \phi\|_{L^2}^2 \|\Delta \phi\|_{L^2}^2}{\|\phi\|_{L^2}^4} &\leq 2 \frac{\|\nabla \phi\|_{L^2}^3 \|\nabla \Delta \phi\|_{L^2}}{\|\phi\|_{L^2}^4} \\ &= 2 \mathcal{E}^{\frac{3}{2}} \frac{\|\nabla \Delta \phi\|_{L^2}}{\|\phi\|_{L^2}} \\ &\leq \frac{1}{3} \frac{\|\nabla \Delta \phi\|_{L^2}^2}{\|\phi\|_{L^2}^2} + \mathcal{E}^3. \end{aligned}$$

Finally we have

$$\begin{aligned} \frac{\|\nabla \phi\|_{L^2}^2 \int_M \left[ \frac{1}{2} \text{tr}_g E \phi^2 \right]}{\|\phi\|_{L^2}^4} &\leq C \frac{\|\nabla \phi\|_{L^2}^2 \|E\|_{L^2} \|\phi\|_{L^4}^2}{\|\phi\|_{L^2}^4} \\ &\leq CA \frac{\mathcal{E} \|E\|_{L^2}}{\|\phi\|_{L^2}^2} (\|\nabla \phi\|_{L^2}^2 + \|\phi\|_{L^2}^2) \\ &\leq CA \|E\|_{L^2} (\mathcal{E}^2 + \mathcal{E}). \end{aligned}$$

Combining these estimates gives the result.  $\square$

*Proof of Theorem 5.1.* Let  $\phi_T$  denote an eigenfunction for the first Dirichlet eigenvalue of  $g_T$ , and let  $\phi_t$ ,  $t \in [0, T]$  denote the solution to (5.1) with  $\phi_T$  as the value at  $t = T$ . Note that  $\int_M \phi_T dV_T = 0$ , and by Lemma 5.3 one has  $\int_M \phi_t dV_t = 0$  for any  $0 \leq t \leq T$ . Also, note that  $\phi_t$  does not vanish identically for any  $t$ . Indeed, if there was a  $t$  such that  $\phi_t \equiv 0$ , we note that  $\phi_s \equiv 0$  is the unique solution to (5.1), forcing  $\phi_T \equiv 0$ , a contradiction. This implies that for any  $t$ ,  $\phi_t$  is a valid test function for estimating  $\lambda(g_t)$ .

Provided  $\bar{\lambda}$  is chosen sufficiently small with respect to universal constants, as long as  $\mathcal{E}(g, \phi) \leq 4\bar{\lambda}$  and  $\epsilon \leq \frac{1}{2A}$  we conclude from Proposition 5.6 that

$$\frac{\partial}{\partial \tau} \mathcal{E}(g, \phi) \leq C \left( A^2 \|E\|_{L^2}^2 + A \|E\|_{L^2} + \bar{\lambda}^2 \right) \mathcal{E} + C \left( A^2 \|E\|_{L^2}^2 + A \|E\|_{L^2} \right)$$

Applying Gronwall's inequality yields, for any  $t \leq T$ , as long as  $\sup_{[t, T]} \mathcal{E}(g, \phi) \leq 2\bar{\lambda}$ ,

$$\mathcal{E}(g_t, \phi_t) \leq \exp \left[ C \int_t^T \left( A^2 \|E\|_{L^2}^2 + A \|E\|_{L^2} + \bar{\lambda}^2 \right) dt \right] \left( \int_t^T \left( A^2 \|E\|_{L^2}^2 + A \|E\|_{L^2} \right) + \mathcal{E}(g_T, \phi_T) \right)$$

Next we can estimate

$$C \int_t^T A^2 \|E\|_{L^2}^2 \leq CA^2 \int_0^T \|E\|_{L^2}^2 \leq CA^2 \epsilon \leq \frac{1}{3} \ln 2$$

for  $\epsilon$  chosen sufficiently small with respect to  $A$  and  $\lambda(g_T)$ . Likewise we have

$$C \int_t^T A \|E\|_{L^2} \leq CA \left( \int_0^T \|E\|_{L^2}^2 \right)^{\frac{1}{2}} \left( \int_0^T 1 \right)^{\frac{1}{2}} \leq CA \epsilon^{\frac{1}{2}} T^{\frac{1}{2}} \leq \frac{1}{3} \ln 2$$

provided  $\epsilon$  is chosen sufficiently small with respect to  $A$  and  $T \leq 1$ . Finally we estimate

$$C \int_t^T \bar{\lambda}^2 \leq CT \bar{\lambda}^2 \leq \frac{1}{3} \ln 2$$

provided  $\bar{\lambda}$  is chosen sufficiently small with respect to universal constants and  $T \leq 1$ . Combining these estimates yields first of all that for any time  $0 \leq t \leq T$ ,

$$\mathcal{E}(g_t, \phi_t) \leq 2 \left( \mathcal{E}(g_T, \phi_T) + CA^2 \epsilon^{\frac{1}{2}} \right) \leq 4\bar{\lambda},$$

i.e. the condition  $\mathcal{E}(g_t, \phi_t) \leq 4\bar{\lambda}$  holds on  $[0, T]$ . Note that this last inequality requires that we choose  $\epsilon$  small with respect to  $A$  and  $\bar{\lambda}$ , but of course  $\bar{\lambda}$  is universal. Hence

$$\lambda(g_0) = \inf_{\{\phi | \int_M \phi dV = 0\}} \frac{\|\nabla \phi\|_{L^2}^2}{\|\phi\|_{L^2}^2} \leq \mathcal{E}(g_0, \phi_0) \leq 2\mathcal{E}(g_T, \phi_T) + CA^2 \epsilon^{\frac{1}{2}} = 2\lambda(g_T) + CA^2 \epsilon^{\frac{1}{2}}$$

as required.  $\square$

## 6. LOW-ENERGY CONVERGENCE ON THREE-MANIFOLDS

In this section we prove Theorem 1.4. First we recall some comparison geometry results for manifolds with supercritical  $L^p$  bounds on curvature.

**Definition 6.1.** Let  $(M^n, g)$  be a Riemannian manifold. Let

$$k(\lambda, p) = \int_M (\max\{0, (n-1)\lambda - \text{Rc}_-\})^p dV.$$

**Theorem 6.2.** ([18] Theorem 1.1) Let  $x \in M$ ,  $\lambda \leq 0$ , and  $p > \frac{n}{2}$  be given, then there is a constant  $C(n, p, \lambda, R)$  which is nondecreasing in  $R$  such that when  $r < R$  we have

$$\left( \frac{\text{Vol } B(x, R)}{v(n, \lambda, R)} \right)^{\frac{1}{2p}} - \left( \frac{\text{Vol } B(x, r)}{v(n, \lambda, r)} \right)^{\frac{1}{2p}} \leq C(n, p, \lambda, R) k(\lambda, p)^{\frac{1}{2p}}.$$

**Theorem 6.3.** ([9] Theorem 3) Let  $\alpha$  and  $D$  be any positive constants and  $p > \frac{n}{2}$ . In any Riemannian manifold  $(M^n, g)$  with  $\text{diam}(g) \leq D$  satisfying

$$\frac{1}{\text{Vol}(M)} \int_M \left( \max \left\{ 0, \frac{\text{Rc}_-}{\alpha^2(n-1)} - 1 \right\} \right)^p dV \leq \frac{1}{2(e^{B(p)\alpha D} - 1)},$$

every domain  $\Omega$  satisfies

$$\frac{\text{Area}(\partial\Omega)}{\text{Vol}(M)} \geq \gamma(\alpha, D) \min \left\{ \frac{\text{Vol}(\Omega)}{\text{Vol}(M)}, \frac{\text{Vol}(M \setminus \Omega)}{\text{Vol}(M)} \right\}^{1-\frac{1}{p}}.$$

**Remark 6.4.** Observe that an  $L^p$  energy bound,  $p > \frac{n}{2}$ , implies an upper bound on the volume growth of balls using Theorem 6.2. Therefore in the presence of such a bound the volume is bounded above in terms of the diameter.



**Corollary 6.5.** *Given  $V > 0, D > 0$  there exists  $\epsilon > 0$  so that if  $(M^3, g)$  is a compact Riemannian manifold with  $\text{Vol}(g) \geq V$ ,  $\text{diam}(g) \leq D$ ,  $\mathcal{F}(g) \leq \epsilon$  then there is a constant  $C = C(V, D)$  such that  $C_S(g) \leq C$ .*

*Proof.* Choose  $\epsilon$  so that

$$\epsilon \leq \frac{1}{2(e^{B(2)D} - 1)},$$

where  $B(2)$  is the constant from Theorem 6.3. Since  $\max\left\{0, \frac{\text{Rc}_-}{n-1} - 1\right\} \leq \frac{|\text{Rc}|}{n-1}$  and  $\text{Vol}(M) \geq V$ , Theorem 6.3 applies with  $\alpha = 1, p = 2$  to conclude that there is a constant  $\gamma$  depending on  $V$  and  $D$ , so that for any subdomain  $\Omega$ ,

$$\frac{\text{Area}(\partial\Omega)}{\min\{\text{Vol}(\Omega), \text{Vol}(M \setminus \Omega)\}^{\frac{2}{3}}} \geq \gamma \frac{1}{\min\{\text{Vol}(\Omega), \text{Vol}(M \setminus \Omega)\}^{\frac{1}{6}}} \geq \gamma$$

The last line follows since there is a uniform upper bound on the volume of  $M$  as observed in Remark 6.4. Thus the isoperimetric constant is bounded, and the result now follows from the discussion in Remark 2.4.  $\square$

*Proof of Theorem 1.4.* Note that since we have assumed a lower bound for the volume, and by Remark 6.4 we have an upper bound for the volume as long as  $\epsilon \leq 1$ , we can rescale to unit volume and it suffices to show the theorem for such metrics. First we aim to show a certain short-time existence statement for solutions to the  $L^2$  flow. We claim that given  $D$  there exists a large constant  $K$ , and small constants  $\epsilon > 0$  and  $T > 0$  such that if  $(M^3, g)$  is a compact Riemannian manifold satisfying

$$(6.1) \quad \text{Vol}(g) = 1, \quad \text{diam}(g) \leq D, \quad \mathcal{F}(g) \leq \epsilon$$

then the solution to the  $L^2$  flow exists on  $[0, T]$  and satisfies the estimates

$$(6.2) \quad \begin{aligned} \sup_{[0, T]} \text{diam}(g_t) &< KD, \\ \sup_{[0, T]} t^{\frac{1}{2}} |\text{Rm}|_{C^0(g_t)} &< 1. \end{aligned}$$

If the claim were false, then for any choice of  $K$ , we obtain sequences  $\epsilon_i \rightarrow 0$ ,  $t_i \rightarrow 0$ , and compact Riemannian manifolds  $(M_i^3, g_i)$  such that  $g_i$  satisfies (6.1) with  $\mathcal{F}(g) \leq \epsilon_i$  and the solution to the  $L^2$  flow with initial condition  $g^i$  satisfies the estimates (6.2) on a maximal time interval  $[0, t_i]$ .

We aim to derive a contradiction from this statement for sufficiently large  $K$ . First we claim that as long as (6.2) holds there is a uniform constant  $A$  depending on  $D$  and  $K$  such that

$$\sup_{[0, t_i]} C_S(g_t^i) \leq A$$

As long as  $\epsilon_i$  is sufficiently small with respect to  $K$  and  $D$ , this follows directly from Corollary 6.5. Suppose now that the second condition of (6.2) failed at time  $t_i$ , i.e.

$$\sup_M |\text{Rm}|_{C^0(g_{t_i}^i)} = t_i^{-\frac{1}{2}}.$$

Define the sequence of time dependent metrics

$$(6.3) \quad \tilde{g}^i(t) = t_i^{-\frac{1}{2}} g^i(t_i \cdot t).$$

The family of metrics  $\tilde{g}^i(t)$  exists on  $[0, 1]$  and

$$\sup_{[\frac{1}{2}, 1]} C_S(\tilde{g}^i) \leq A,$$

$$\sup_{[\frac{1}{2}, 1]} \left| \widetilde{\text{Rm}}^i \right|_{C^0(\tilde{g}^i)} < 2.$$

Moreover, by construction  $\left| \widetilde{\text{Rm}}^i \right|_{C^0(\tilde{g}_1^i)} = 1$ , and we let  $x^i \in M^i$  be a point realizing this supremum. Using the bound on the Sobolev constant, one has a lower bound for the volume growth of small balls (see Lemma 2.6), so Cheeger's lemma implies that  $\text{inj}_{\tilde{g}^i} \geq \nu > 0$  for some small constant  $\nu$ . By ([20] Theorem 7.1, see also [24] Corollary 1.6) the sequence  $\{(M^i, \tilde{g}_t^i, x^i)\}$  contains a subsequence converging to  $(M^\infty, g_t^\infty, x^\infty)$ . Moreover, one has  $|\text{Rm}^\infty|_{g_1^\infty}(x^\infty) = 1$ . However, since  $\epsilon_i \rightarrow 0$ , one has  $\mathcal{F}(\tilde{g}_1^i) \rightarrow 0$ . By Fatou's Lemma we can conclude  $\mathcal{F}(g_1^\infty) = 0$ , contradicting nonflatness of  $(M^\infty, g^\infty, x^\infty)$ . Thus this possibility is ruled out.

Therefore it must be the case that the first condition of (6.2) fails. We will work with one element of the sequence and drop the index  $i$  from the notation. We want to derive a contradiction by showing that the first Dirichlet eigenvalue of  $(M, g(t_i))$  is quite small, then using Theorem 5.1 to show that the initial Dirichlet eigenvalue had to be quite small, a contradiction. We will estimate  $\lambda(g_{t_i})$  using the trick that

$$\lambda(M) \leq \max\{\mu(M_1), \mu(M_2)\}$$

where  $M_i$  are disjoint open subsets of  $M$  and  $\mu(M_i)$  denotes the first Dirichlet eigenvalue. To that effect, since  $\text{diam}(g_{t_i}) = KD$  we choose points  $x, y$  such that  $d_{g_{t_i}}(x, y) = KD$  and estimate  $\mu\left(B_{\frac{KD}{2}}(x)\right)$  above. To simplify notation let  $R = \frac{KD}{2}$ , and let  $\phi \in C^1(M)$  satisfy

$$\phi|_{B_{\frac{R}{2}}(x)} \equiv 1, \quad \text{supp } \phi \subset B_R(x), \quad |\nabla \phi| \leq \frac{C}{R}.$$

Observe that  $\phi$  is nonconstant and moreover

$$\int_M |\nabla \phi|^2 \leq \frac{C}{R^2} \text{Vol}(B_R(x_i)), \quad \int_M \phi^2 \geq \text{Vol}(B_{\frac{R}{2}}(x_i)).$$

However, since the Sobolev constant is bounded, there is a certain constant  $\mu = \mu(K, D)$  such that

$$\text{Vol}(B_R(x_i)) \geq \mu(K, D).$$

Applying Theorem 6.2 with  $r = \frac{R}{2}$  and  $\lambda = 0$ , we observe that

$$\text{Vol}(B_{\frac{R}{2}}(x)) \geq \left( \text{Vol}(B_R(x))^{\frac{1}{2}} - C(n, R)k(\lambda, p)^{\frac{1}{2}} \right)^2.$$

If we choose  $\epsilon$  small enough so that

$$C(n, R)k(\lambda, p) \leq \frac{1}{2}\mu$$

we may conclude that

$$\text{Vol}(B_{\frac{R}{2}}) \geq \delta \text{Vol}(B_R(x))$$

for some universal constant  $\delta$ . We conclude that for a new constant  $C$  one has

$$\mu\left(B_{\frac{R}{2}}(x)\right) \leq \frac{\|\nabla \phi\|_{L^2}^2}{\|\phi\|_{L^2}^2} \leq \frac{C}{2K^2D^2}.$$

One estimates the first Dirichlet eigenvalue of the ball of radius  $\frac{R}{2}$  around  $y$  identically, and thus we yield

$$\lambda(g_{t_i}) \leq \frac{C}{2K^2D^2}.$$

Now suppose that  $K$  is sufficiently large that  $\frac{C}{K^2D^2} \leq \bar{\lambda}$ , where  $\bar{\lambda}$  is the constant from Theorem 5.1. Also note that since at each time the metric has bounded volume and bounded Sobolev constant  $L^6 \rightarrow H_1^2$ , a simple application of Hölder's inequality shows that the constant  $A$  of Theorem 5.1 is bounded in terms of the given Sobolev constant. We now choose  $\epsilon$  sufficiently small with respect to this bound (which depends on  $K$  and  $D$ ), so that Theorem 5.1 applies to conclude

$$\lambda(g_0) \leq 2\lambda(g_{t_i}) + CA^2\epsilon^{\frac{1}{2}} \leq \frac{C}{K^2D^2}.$$

However, from Theorem 6.3 we know that if  $\epsilon$  is chosen sufficiently small there is a lower bound on the isoperimetric ratio

$$\frac{\text{Area}(\partial\Omega)}{\min\{\text{Vol}(\Omega), \text{Vol}(M \setminus \Omega)\}} \geq \gamma(D) \text{Vol}(M)^{\frac{1}{2}} \min\{\text{Vol}(\Omega), \text{Vol}(M \setminus \Omega)\}^{-\frac{1}{2}}.$$

Since  $\text{Vol}(g_0) = 1$ , we conclude

$$h(M, g_0) := \int_{\Omega \subset M} \frac{\text{Area}(\partial\Omega)}{\min\{\text{Vol}(\Omega), \text{Vol}(M \setminus \Omega)\}} \geq \gamma(D).$$

By Cheeger's inequality [7] we conclude

$$\lambda(g_0) \geq \frac{h(M, g_0)^2}{4} \geq \frac{\gamma(D)^2}{4}.$$

Choosing  $K$  sufficiently large with respect to  $\gamma(D)$ , we may conclude

$$\frac{\gamma(D)^2}{4} \leq \lambda(g_0) \leq \frac{C}{K^2D^2} < \frac{\gamma(D)^2}{4},$$

a contradiction. Thus the claim of uniform short time existence follows.

To finish the proof we use a version of the implicit function theorem for solutions to the  $L^2$  flow near flat manifolds. In particular, we continue arguing by contradiction, and given  $D > 0$  we assume that for every  $\epsilon > 0$  there is a three-manifold satisfying the hypotheses of the theorem but for which the flow does not exist for all time and converge to a flat metric. Choose a sequence  $\epsilon_i \rightarrow 0$  and  $(M_i^3, g_i)$  realizing this possibility. By the discussion above, for sufficiently small  $\epsilon_i$  we have that the solution to the  $L^2$  flow exists on  $[0, T]$ , and moreover

$$\begin{aligned} \text{diam}(g_T^i) &< KD \\ \left| \nabla^k \text{Rm} \right|_{C^0(g_T^i)} &\leq C_k. \end{aligned}$$

By the discussion above, we also conclude a uniform lower bound on the injectivity radius of  $g_T^i$ . It follows from [11] Theorem 2.3 that we may take a subsequence of  $\{(M_i^3, g_T^i)\}$  which converges in the  $C^k$  topology for any  $k$ , necessarily to a flat metric. At this point one can repeat the argument of ([21] Theorem 1.6) to conclude that for  $g_T^i$  sufficiently close to a flat metric in  $C^k$ , the  $L^2$  flow exists for all time and converges exponentially to a flat metric. Given this exponential convergence, it is a straightforward matter to show that the volume normalized  $L^2$  flow also exists for all time and converges to a flat metric.  $\square$

**Corollary 6.6.** *Given  $V > 0, D > 0$  there exists  $\epsilon > 0$  sufficiently small so that the space of metrics on  $T^3$  satisfying  $\text{Vol}(g) \geq V$ ,  $\text{diam}(g) \leq D$ ,  $\mathcal{F}(g) \leq \epsilon$  is connected in the  $C^\infty$  topology.*

*Proof.* Theorem 1.4 guarantees that for  $\epsilon > 0$  sufficiently small, metrics on  $T^3$  satisfying  $\text{Vol}(g) \geq V$ ,  $\text{diam}(g) \leq D$  and  $\mathcal{F}(g) \leq \epsilon$  are smoothly deformable to flat metrics. Since the space of flat metrics on  $T^3$  is path-connected, the corollary follows.  $\square$

## 7. LOW ENERGY CONVERGENCE ON FOUR-MANIFOLDS

In this section we investigate the  $L^2$  flow with low energy on four-manifolds. The optimal convergence result in this direction would be an analogue of Theorem 1.4, i.e. given energy sufficiently small with respect to the Sobolev constant, the solution to the  $L^2$  flow exists for all time and converges to a flat metric. The first test of this claim is to determine if there are any other critical points of  $\mathcal{F}$  in this regime.

**Proposition 7.1.** *Given  $A > 0$ , there exists  $\epsilon > 0$  so that if  $(M^4, g)$  is a compact Riemannian manifold satisfying*

$$\begin{aligned} \text{grad } \mathcal{F} &\equiv 0 \\ C_S(g) &\leq A \\ \mathcal{F}(g) &\leq \epsilon, \end{aligned}$$

*then  $g$  is flat.*

*Proof.* If the statement was false, then given  $A > 0$ , there exists a sequence  $\epsilon_i \rightarrow 0$  and a sequence of compact Riemannian manifolds  $\{(M_i^4, g^i)\}$  of compact critical, nonflat four-manifolds satisfying the hypotheses of the theorem. By rescaling, we may assume without loss of generality that the metrics satisfy  $\text{Vol}(g^i) = 1$ . We first claim is that there is a uniform curvature bound along the sequence. If not, there is some subsequence such that

$$K_i := |\text{Rm}|_{g^i}(x_i) = |\text{Rm}|_{C^0(g^i)} \rightarrow \infty$$

The sequence of manifolds  $\{(M_i^4, K_i g^i, x_i)\}$  has uniform bounds on all covariant derivatives of curvature, and so converges to a noncompact, nonflat critical four-manifold. But since  $\epsilon_i \rightarrow 0$  it follows that this limiting manifold must be flat, a contradiction.

Since there is a uniform bound on the curvature along the sequence, there are also uniform bounds on all higher derivatives of curvature. Also, since the Sobolev constants are bounded, we obtain a uniform lower bound on the injectivity radius of  $g^i$ , and a uniform upper bound on the diameter. It follows that we may take a limit of  $\{(M_i^4, g^i)\}$ , which is necessarily flat. In particular, for large enough  $i$   $g^i$  is  $C^k$ -close to a flat metric for arbitrary  $k$ . It follows from ([21] Theorem 1.6) that the solution to the  $L^2$  flow with initial condition  $g^i$  exists for all time and converges to a flat metric. But since  $g^i$  is critical, the flow is stationary, therefore  $g^i$  is already flat, a contradiction. The proposition follows.  $\square$

Next we give the proof of Theorem 1.6, which says that, in determining the behavior of the  $L^2$  flow on four-manifolds with energy small with respect to the Sobolev constant, the problems lie in understanding the short-time behavior of the Sobolev constant.

*Proof of Theorem 1.6.* We begin by showing a certain uniform short time existence statement. In particular, we claim that we may choose  $\epsilon(A, B)$  so that if  $(M^4, g_0)$  is as in the statement of the theorem, then the solution to the  $L^2$  flow exists on  $[0, 1]$  and moreover satisfies the estimates

$$(7.1) \quad \begin{aligned} \sup_{t \in [0, 1]} C_S(g_t) &< 2B, \\ t^{\frac{1}{2}} |\text{Rm}|_{C^0(g_t)} &< 1. \end{aligned}$$

If this is false, then we have a sequence  $\epsilon_i \rightarrow 0$  and a sequence of compact Riemannian four-manifolds  $\{(M^i, g^i)\}$  satisfying  $C_S(g^i) \leq B$  and  $\mathcal{F}(g^i) \leq \epsilon_i$ , such that if  $g_t^i$  denotes the

solution to the  $L^2$  flow with initial condition  $g^i$ , one the estimates of (7.1) fails at some time  $t_i < 1$ . Suppose there existed a subsequence where the second condition of (7.1) failed at  $t_i$ , i.e.  $t_i^{\frac{1}{2}} |\text{Rm}|_{C^0(g_{t_i}^i)} = 1$ . Define

$$\tilde{g}^i(t) = t_i^{-\frac{1}{2}} g^i(t_i \cdot t).$$

Each one-parameter family  $\tilde{g}^i(t)$  exists on  $[0, 1]$  and moreover

$$\sup_{[0,1]} C_S(\tilde{g}^i) < 2B, \quad \sup_{[\frac{1}{2},1]} \left| \widetilde{\text{Rm}}^i \right|_{C^0(\tilde{g}^i)} < 2.$$

Moreover, by construction  $\left| \widetilde{\text{Rm}}^i \right|_{C^0(\tilde{g}_1^i)} = 1$ , and we let  $x^i$  be a point realizing this supremum.

Using the bound on  $C_S(\tilde{g}^i)$ , one automatically obtains a scale-invariant lower bound on the volume growth of balls, and then it follows from Cheeger's lemma that  $\text{inj}_{\tilde{g}^i} \geq \nu > 0$  for some small constant  $\nu$ . By ([20] Theorem 7.1, see also [24] Corollary 1.6) the sequence  $(M^i, \tilde{g}_t^i, x^i)$  contains a subsequence converging to  $(M^\infty, g_t^\infty, x^\infty)$ . Moreover, one has  $|\text{Rm}^\infty|(x^\infty) = 1$ . However, since  $\epsilon_i \rightarrow 0$ , one has  $\mathcal{F}(\tilde{g}_1^i) \rightarrow 0$ . By Fatou's Lemma we can conclude  $\mathcal{F}(g_1^\infty) = 0$ , contradicting nonflatness of  $(M^\infty, g^\infty, x^\infty)$ . Thus this possibility is ruled out.

Now using the curvature decay we will show that the first condition of (7.1) holds on  $[0, 1]$  for  $\epsilon$  sufficiently small. In particular, we first note from Theorem 3.4 that on  $[0, 1]$  we may conclude uniform estimates

$$(7.2) \quad t^{\frac{k+2}{4}} \left| \nabla^k \text{Rm} \right| < C_k.$$

Furthermore, we may choose  $\epsilon$  small with respect to  $B$  such that, as long as (7.1) holds, we have

$$\sup_{[0,T]} \|\text{grad } \mathcal{F}\|_{L^2}^2 \leq 2A.$$

Now applying Theorem 2.5 with  $m = 2$ ,  $q = 6$ ,  $p = \infty$  and  $\alpha = \frac{3}{4}$ , we conclude that, as long as (7.1) holds,

$$(7.3) \quad \int_0^T \|\text{grad } \mathcal{F}\|_\infty \leq CB^{\frac{3}{4}} A^{\frac{1}{4}} \int_0^T \left( \|\nabla \text{grad } \mathcal{F}\|_{L^q}^{\frac{3}{4}} + \|\text{grad } \mathcal{F}\|_{L^q}^{\frac{3}{4}} \right) dt.$$

Using (7.2) and the fact that  $\text{Vol}(g) = 1$ , we conclude that

$$\begin{aligned} \|\nabla \text{grad } \mathcal{F}\|_{L^q} &\leq C \text{Vol}^{\frac{1}{q}} (\|\nabla^3 \text{Rm}\|_\infty + \|\nabla \text{Rm}\|_\infty \|\text{Rm}\|_\infty) \\ &\leq Ct^{-\frac{5}{4}}. \end{aligned}$$

Similarly

$$\|\text{grad } \mathcal{F}\|_{L^q} \leq Ct^{-1}.$$

Plugging these into (7.3) yields, as long as  $T \leq 1$ ,

$$\int_0^T \|\text{grad } \mathcal{F}\|_\infty \leq CB^{\frac{3}{4}} A^{\frac{1}{4}} T^{\frac{1}{16}}.$$

In particular, given  $\delta > 0$  we may choose  $T$  sufficiently small with respect to  $A$  and  $B$  so that, for all  $t \in [0, T]$ ,

$$(1 + \delta)^{-1} g_0 \leq g_t \leq (1 + \delta) g.$$

Furthermore for  $\delta$  chosen sufficiently small with respect to universal constants this implies

$$C_S(g_t) \leq \frac{3}{2}C_S(g_0),$$

and the short time existence claim is finished.

To finish the proof we apply an analytic stability result for the  $L^2$  flow near flat metrics. It follows from Theorem 3.4 that for each  $k$  one has uniform estimates on  $|\nabla^k \text{Rm}|_{C^0(g_1^i)}$ . Furthermore, since  $\text{Vol}(g_1^i) = 1$ , from Lemma 2.6 we have a uniform upper bound on  $\text{diam}(g_1^i)$ , and from (2.1) and Cheeger's lemma a uniform lower bound on  $\text{inj}(g_1^i)$ . It follows from ([11] Theorem 2.3) that there exists a subsequence of  $\{g_1^i\}$  converging in any  $C^k$  norm to a flat metric. It follows from ([21] Theorem 1.6) that for sufficiently large  $i$ , the solution to  $L^2$  flow with initial condition  $g_1^i$  exists for all time and converges exponentially to a flat metric. This finishes the proof.  $\square$

## 8. HIGHER DIMENSIONS

We first observe a simple proposition which exemplifies the role of the dimension in understanding solutions to the  $L^2$  flow. In particular, we note that finite time singularities certainly occur in dimensions  $n \geq 5$ .

**Proposition 8.1.** *Consider  $(S^n, g_{S^n})$  where  $g_{S^n}$  is the metric of constant sectional curvature  $K \equiv 1$ . The solution to (1.2) with initial condition  $g_{S^n}$  exists*

- on  $[0, \infty)$  and satisfies  $g(t) = c_n(1 + \sqrt{t})g_{S^n}$  for  $n = 2, 3$ .
- on  $[0, \infty)$  and satisfies  $g(t) = g_{S^n}$ .
- on  $[0, \frac{1}{c_n})$  and satisfies  $g(t) = \sqrt{1 - c_n t}g_{S^n}$ ,

where  $c_n$  is a constant depending on the dimension.

*Proof.* The metric  $g_{S^n}$  satisfies

$$\nabla \text{Rc}_{g_{S^n}} = 0, \quad \check{R}_{g_{S^n}} = \frac{1}{n} |\text{Rm}|^2 g = \frac{1}{n} (n(n-1)) g$$

It follows that the solution to the  $L^2$  flow with initial condition  $Ag_{S^n}$  reduces to the ODE

$$\frac{\partial}{\partial t} A = \frac{(\frac{1}{n} - \frac{1}{4}) n(n-1)}{A}.$$

The proposition follows immediately.  $\square$

As it turns out, not only does the  $L^2$  flow encounter finite time singularities in dimension  $n \geq 5$ , in general they need not satisfy a noncollapsing estimate. We next recall Perelman's no local collapsing result for Ricci flow. First we recall the definition of  $\kappa$ -collapsing on a given scale.

**Definition 8.2.** A Riemannian manifold  $(M^n, g)$  is said to be  $\kappa$ -collapsed at the scale  $r$  if there exists  $x \in M$  such that  $|\text{Rm}| \leq r^{-2}$  for all  $x \in B(x, r)$ , and

$$\frac{\text{Vol}(B(x, r))}{r^n} \leq \kappa.$$

**Theorem 8.3.** (Perelman) *Let  $g(t), t \in [0, T)$  be a smooth solution to the Ricci flow on a closed manifold  $M^n$ . If  $T < \infty$ , then for any  $\rho \in (0, \infty)$  there exists  $\kappa = \kappa(g(0), T, \rho)$  such that  $g(t)$  is  $\kappa$ -noncollapsed below the scale  $\rho$  for all  $t \in [0, T)$ .*

This theorem has a corollary fundamental to the analysis of finite time singularities of Ricci flow.

**Corollary 8.4.** *Let  $(M^n, g(t)), t \in [0, T], T < \infty$  be a solution to the Ricci flow on a closed manifold. For every  $C > 0$  there exists  $\alpha > 0$  depending on  $C, g(0)$ , and  $T$  such that if  $(x, t)$  satisfies*

$$|\text{Rm}|_{g(t)} \leq CK$$

*on  $B_{g(t)}\left(x, \frac{1}{\sqrt{CK}}\right)$ , where  $K = |\text{Rm}|_{g(x,t)}$ , then*

$$\text{inj}_{g(t)}(x) \geq \frac{\alpha}{\sqrt{K}}.$$

Alas, in high dimensions it is possible for solutions to the  $L^2$  flow to fail to satisfy an injectivity radius estimate on the scale of maximum curvature.

**Proposition 8.5.** *Consider  $M^5 = S^5 \times S^1$ , and let  $g_0 = A_0 g_{S^5} \oplus B_0 g_{S^1}$ , where  $g_{S^n}$  denotes the metric of constant sectional curvature  $K \equiv 1$ . The solution to the  $L^2$  flow with this initial condition exists on a finite time interval  $[0, T]$ , and moreover,*

$$\lim_{t \rightarrow T} |\text{Rm}|_{g_t} \text{inj}_{g_t}^2 = 0.$$

*Proof.* By the uniqueness of solutions to the  $L^2$  flow, the isometry group of  $g_0$  is preserved along the flow. In particular, the flow will reduce to an ODE on the parameters  $A$  and  $B$ , i.e. we may express

$$g_t = A_t g_{S^5} \oplus B_t g_{S^1}.$$

The curvature tensor of any such metric is parallel, therefore  $\delta d \text{Rc} \equiv 0$  along the flow. Next define the dimensional constant  $c_n := |\text{Rm}(g_{S^n})|_{g_{S^n}}^2$ . It follows that

$$|\text{Rm}|^2 g = \frac{c_5}{A^2} (A g_{S^5} \oplus B g_{S^1})$$

Next, since  $g_{S^5}$  has constant curvature and  $g_{S^1}$  is flat, one can check that  $\check{R}$  must be a multiple of  $g_{S^5}$ . Using that  $\text{tr}_g \check{R} = |\text{Rm}|^2$ , it follows that

$$\check{R} = \frac{c_5}{5A} g_{S^5}.$$

It follows that the solution to the  $L^2$  flow is reduced to the system of ODEs

$$\begin{aligned} \frac{\partial}{\partial t} A &= \frac{c_5}{5A} - \frac{c_5}{4A} = -\frac{c_5}{20A} \\ \frac{\partial}{\partial t} B &= -\frac{c_5 B}{4A^2}. \end{aligned}$$

The solution exists on  $\left[0, \frac{20A_0^2}{c_5}\right]$ , and one has

$$A_t = \sqrt{2A_0^2 - \frac{tc_5}{10}}.$$



We have shown the existence statement, next we show that  $|\text{Rm}|\text{inj}^2$  approaches zero at the singular time. One directly computes that

$$\begin{aligned} \frac{\partial}{\partial t} \ln \left( \frac{B}{A^p} \right) &= \frac{\partial}{\partial t} (\ln B - p \ln A) \\ &= \frac{\frac{\partial}{\partial t} B}{B} - p \frac{\frac{\partial}{\partial t} A}{A} \\ &= -\frac{c_5}{4A^2} - p \left( -\frac{c_5}{20A^2} \right) \\ &= \frac{c_5}{4A^2} \left( \frac{p}{5} - 1 \right). \end{aligned}$$

Therefore we have that

$$\frac{B_t}{A_t^5} = \frac{B_0}{A_0^5}.$$

Now let  $\theta$  denote the standard coordinate on  $S^1$ . For any  $x \in S^5$ , since the metric is a Riemannian product, the lateral curve  $\gamma(\theta) = (x, \theta)$  is a geodesic. Its length is  $2\pi\sqrt{B}$ , and is not minimizing past length  $\pi\sqrt{B}$ . Thus  $\text{inj}_{g_t} \leq \pi\sqrt{B_t}$ . Let  $T = \frac{20A_0^2}{c_5}$ . Since  $|\text{Rm}|_{g_t}^2 = \frac{c_5}{A_t^2}$  we thus have that

$$\lim_{t \rightarrow T} |\text{Rm}|_{g_t} \text{inj}_{g_t}^2 \leq \lim_{t \rightarrow T} \frac{c_5}{A_t} \pi B_t = \lim_{t \rightarrow T} \frac{\pi c_5 B_0}{A_0^5} A_t^4 = 0.$$

□

**Remark 8.6.** Note that this example shows that this behavior can occur in any dimension  $n \geq 6$ , by simply taking a product with a torus of arbitrary dimension. In all likelihood one can find an example in dimension  $n = 5$  which experiences collapse at the scale of maximum curvature, but so far no easy example presents itself. Therefore, any effort to show that finite time singularities of the  $L^2$  flow are not collapsed at the scale of curvature must take the dimension into account in some identifiable way.

## 9. CONJECTURAL FRAMEWORK

The gradient for of the functional  $\mathcal{F}(g)$  can be thought of as an intrinsic Riemannian analogue of the Yang-Mills energy. Observing the scaling law  $\mathcal{F}(\lambda g) = \lambda^{\frac{n}{2}-2} \mathcal{F}(g)$ , one can hope for good regularity properties of the gradient flow in dimensions  $n = 2, 3$ , and dimension  $n = 4$  with sufficiently small energy. Let us recall some results from the theory of Yang-Mills flow which illustrate this behavior.

**Theorem 9.1.** (*Rade [19]*) *Let  $(M^n, g)$  be a compact Riemannian manifold with  $n = 2, 3$ . Let  $E \rightarrow M$  denote the total space of a vector bundle over  $M$  with semisimple structure group. If  $A_0$  denotes a connection on  $E$ , the solution to the Yang-Mills flow with initial condition  $A_0$  exists for all time and converges to a Yang-Mills connection.*

**Remark 9.2.** The proof is via Moser iteration, where the supercriticality of the functional exhibits itself in a clear fashion. An a-priori bound on the Sobolev constant of the base manifold is essential to this proof, and as we have remarked above it is precisely this lack of a-priori control over the Sobolev constant which provides such extreme difficulty in understanding solutions to the  $L^2$  flow. Furthermore, the issue of convergence is not immediately settled by this proof as the estimates degenerate at infinite time.

**Theorem 9.3.** (*Struwe [25]*) *Let  $(M^4, g)$  be a compact Riemannian manifold, and let  $E \rightarrow M$  denote the total space of a vector bundle over  $M$  with semisimple structure group. Let  $A_0$  denote*

a connection on  $E$ . The solution to the Yang-Mills flow with initial condition  $A_0$  exists on a maximal time interval  $[0, T)$ , and

$$T = \sup \left\{ \bar{t} > 0 \mid \exists R > 0, \sup_{x_0 \in M, 0 \leq t \leq \bar{t}} \left( \int_{B_R(x_0)} |F(t)|^2 dV \right) < \epsilon_0 \right\}$$

where  $\epsilon_0 = \epsilon_0(E) > 0$ .

**Remark 9.4.** Struwe proves more than this, and one should consult [25] for the precise result. Observe that one consequence is that if the initial global energy is sufficiently small the flow will exist for all time.

With these results as guideposts, we can make a natural conjecture:

**Conjecture 9.5.** (*Main existence conjecture*): Let  $(M^n, g)$  be a compact Riemannian manifold and suppose either

- $n = 2, 3$ , or
- $n = 4$  and  $\|\text{Rm}\|_{L^2} \leq \epsilon$ ,

where  $\epsilon$  is some universal constant. Then the solution to the  $L^2$  flow exists for all time.

**Remark 9.6.** Certainly one cannot expect convergence at  $t = \infty$  for  $n = 3, 4$ , as solutions in general will collapse.

**Remark 9.7.** The case  $n = 2$  of Conjecture 9.5 was established in [23]. While it is natural to expect convergence of the flow to a constant scalar curvature metric, this is not yet known in general.

Observe that the  $n = 4$  conjecture is actually stronger than the directly analogous statement of Theorem 9.3. In particular, we have asked that the constant  $\epsilon$  be independent of the underlying topology, let alone the initial metric. With this in mind, a certain weaker conjecture when  $n = 4$  may be more attainable.

**Conjecture 9.8.** Given  $C > 0$  there exists  $\epsilon(C) > 0$  so that if  $(M^4, g)$  is a compact Riemannian manifold with  $C_S(g) \leq C$  and  $\mathcal{F}(g) \leq \epsilon$ , the solution to the  $L^2$  flow exists for all time and converges to a flat metric.

One cannot help but wonder if these conjectures provide a path towards resolving an old question of Gromov:

**Conjecture 9.9.** (*Gromov*) There exists  $\epsilon > 0$  so that if  $(M^4, g)$  satisfies  $\|\text{Rm}\|_{L^2} \leq \epsilon$ , then  $M$  admits an  $\mathcal{F}$ -structure.

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